REMARKS ON NEHARI'S PROBLEM, MATRIX A_2 CONDITION, AND WEIGHTED BOUNDED MEAN OSCILLATION

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ABSTRACT. We consider Nehari's problem in the case of non-uniqueness of solution. The solution set is then parametrized by the unit ball of H^∞ by means of so-called *regular generators* — bounded holomorphic functions ϕ. The definition of *regularity* is given below, but let us mention now that 1) the following assumption on modulus of ϕ is sufficient for *regularity*: $\frac{1}{1-|\phi|^2} \in L^1(\mathbb{T})$; 2) there is no necessary and sufficient condition of *regularity* on bounded holomorphic ϕ in terms of $|\phi|$ on $\mathbb{T}$, [12]. This makes reasonable the attempt to find a weaker sufficient condition on $|\phi|$ than the condition in 1). This is done here. Also we are discussing certain new necessary and sufficient conditions of *regularity* in terms of bounded mean (weighted) oscillations of ϕ. They involve the matrix A_2 condition from [21].

1. INTRODUCTION

Recent developments in the inverse scattering/spectral theory [23, 22, 9, 15, 17, 18, 19] stimulated our interest to an old question on the description of the Nehari problem solutions set (precisely the question is formulated in Problem 1.1 below). The Nehari problem is strongly related to the Nonlinear Fourier Analysis [22], or what is basically the same, to the inverse scattering problem for CMV matrices [14].

Here we consider L^p spaces of functions on the unit circle $\mathbb{T}$ and their Hardy subspaces H^p. Recall that the famous Nehari Theorem describes projections of functions of the unite ball of L^∞ onto the Hardy space H^2_-, see e.g. textbooks [16, 10]. Let P_- be the Riesz projector $P_- : L^2 \to H^2_-$. The function $F_- \in H^2_-$ possesses the representation

$$F_- = P_- f, \quad \|f\|_\infty \le 1 \tag{1.1}$$

if and only if the corresponding Hankel operator

$$\Gamma x := P_-(F_- x), \quad x \in H^2, \tag{1.2}$$

has norm less or equal to one, $\|\Gamma\| \le 1$ (the operator is naturally defined, say, on polynomials and then extended by continuity) .

Let

$$\mathcal{N}(F_-) = \{f \in L^\infty : F_- = P_- f, \ \|f\|_\infty \le 1\}. \tag{1.3}$$

The Nehari *problem* deals with a description of $\mathcal{N}(F_-)$ for the given F_-. Thus the Nehari Theorem is the solvability condition for this problem. The problem was solved by Adamyan, Arov, and Krein [1, 2, 3]. In the case of non uniqueness the

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set of solutions is parameterized by the unite ball of the class H^∞. Precisely, there exists $\phi = \phi_{F_-} \in H^\infty$ with the following three properties

$$\|\phi\|_\infty \le 1, \quad \int_{\mathbb{T}} \log(1-|\phi|)\,dm > -\infty, \quad \phi(0)=0. \tag{1.4}$$

This function is accompanied by the outer function ψ

$$\psi(\zeta) = e^{\frac{1}{2}\int_{\mathbb{T}} \frac{t+\zeta}{t-\zeta}\log(1-|\phi(t)|^2)\,dm(t)}, \tag{1.5}$$

and the function

$$f_0 = -\frac{\bar\phi\psi}{\bar\psi}. \tag{1.6}$$

The set $\mathcal{N}(F_-)$ is of the form

$$\mathcal{N}(F_-) = \{f = f_{\mathcal{E}} = f_0 + \frac{\psi^2\mathcal{E}}{1-\phi\mathcal{E}} : \mathcal{E}\in H^\infty,\ \|\mathcal{E}\|_\infty \le 1\}. \tag{1.7}$$

This is the hard

Problem 1.1. Specify analytic properties of those holomorphic ϕ's of (1.4) that generate the description (1.7). Following to Arov [4] we call such ϕ's regular.

Remark 1.2. It is convenient to associate with a function ϕ of the form (1.4) the unitary valued matrix function

$$\mathcal{S} = \mathcal{S}_\phi = \begin{bmatrix} \phi & \psi \\ \psi & f_0 \end{bmatrix} \tag{1.8}$$

with the entries given by (1.5), (1.6). Then the relation (1.7) between f and $\mathcal{E}$ can be rewritten into the vector form

$$\begin{bmatrix} A \\ f \end{bmatrix} = \mathcal{S}\begin{bmatrix} A\mathcal{E} \\ 1 \end{bmatrix} = \begin{bmatrix} \phi & \psi \\ \psi & f_0 \end{bmatrix}\begin{bmatrix} A\mathcal{E} \\ 1 \end{bmatrix},$$

where A is defined by this relation in a unique way, $A = \frac{\psi}{1-\phi\mathcal{E}}$. The fact that $\mathcal{S}$ is unitary implies that

$$|f|^2 + |A|^2 = 1 + |A|^2|\mathcal{E}|^2,$$

i.e.:

$$1-|f|^2 = |A|^2(1-|\mathcal{E}|^2) \ge 0.$$

Let us give an example of non-regular ϕ from (1.4). Choose any inner function $\Delta, \Delta(0)>0$. The point is, that a *holomorphic* matrix function

$$\begin{bmatrix} \frac{\Delta-\Delta(0)}{1+\Delta(0)} & \sqrt{\Delta(0)}\frac{1+\Delta}{1+\Delta(0)} \\ \sqrt{\Delta(0)}\frac{1+\Delta}{1+\Delta(0)} & \frac{\Delta-\Delta(0)}{1+\Delta(0)} \end{bmatrix}, \tag{1.9}$$

is unitary on $\mathbb{T}$. That is, for such $\phi = \frac{\Delta-\Delta(0)}{1+\Delta(0)}$ the corresponding $f_0 = \phi$ belongs to H^∞, and thus the class

$$\{f = f_0 + \frac{\psi^2\mathcal{E}}{1-\phi\mathcal{E}} : \mathcal{E}\in H^\infty,\ \|\mathcal{E}\|_\infty \le 1\} \tag{1.10}$$

describes a *proper* subclass of the set $\mathcal{N}(0)$=unit ball of H^∞ (since $P_- f_0 = 0$ in this case).

Let us show that, for instance $f = 0$ can not be represented in this way. First let us note that $\frac{\psi}{1-\phi\mathcal{E}} \in H^2$. In fact, since $\frac{1+\phi\mathcal{E}}{1-\phi\mathcal{E}}$ is a function in the unite disc with the positive real part we have (in the sense of the boundary values on the unite circle)

$$\left|\frac{\psi}{1-\phi\mathcal{E}}\right|^2 \le \frac{1-|\phi\mathcal{E}|^2}{|1-\phi\mathcal{E}|^2} = \mathrm{Re}\frac{1+\phi\mathcal{E}}{1-\phi\mathcal{E}} \in L^1, \tag{1.11}$$

and in addition $\frac{\psi}{1-\phi\mathcal{E}}$ is a function of the Smirnov class (the denominator is an outer function). So if 0 is in the set, we get

$$\frac{f_0}{\psi} = -\frac{\psi\mathcal{E}}{1-\phi\mathcal{E}} \in H^2.$$

On the other hand this function belongs to H^2_- due to the representation

$$\frac{f_0}{\psi} = -\frac{\bar\phi}{\bar\psi}$$

(ϕ/ψ is also of the Smirnov class). Thus $\phi = 0$ and $\psi = 1$.

Note that actually this is a general obstacle: according to the Arov's Theorem one can always "factor out" in a certain sense a holomorphic $\mathcal{S}$-matrix from the given one, so that the remaining part, indeed, generate the description of a Nehari problem in the form (1.7) (the, so called, singular-regular factorization [4]).

On a ceratin stage the answers to Problem 1.1 and to a comparably long list of similar problems (see for instance [7, 8] where the similar question with respect to the Hamburger moment problem is discussed) were formulated in terms of density of a certain class of holomorphic function in an associated with the data Hilbert space.

We need to recall the Nagy–Foias functional model space [20]. It can be associated with an arbitrary function ϕ of the unite ball of H^∞ (the Schur class)

$$K_\phi := H^2 \oplus \overline{\Delta L^2} \ominus \{\phi \oplus \Delta\} H^2,$$

where $\Delta := \sqrt{1-|\phi|^2}$, and

$$\overline{\Delta L^2} = \mathrm{clos}_{L^2}\{f = \Delta g : g \in L^2\}.$$

In our specific case $\log(1-|\phi|^2) \in L^1$ we can chose an "analytic" square root instead of "arithmetic", i.e., to use ψ instead of Δ, and of course $\overline{\Delta L^2} = L^2$. So, the functional space is of the form

$$K_\phi := \begin{bmatrix} H^2 \\ L^2 \end{bmatrix} \ominus \begin{bmatrix} \phi \\ \psi \end{bmatrix} H^2 = \begin{bmatrix} 0 \\ H^2_- \end{bmatrix} \oplus \hat H,$$

where

$$\hat H := \hat H_\phi = H^2(\mathbb{C}^2) \ominus \begin{bmatrix} \phi \\ \psi \end{bmatrix} H^2. \tag{1.12}$$

That is, we have $\begin{bmatrix} x_+ \\ g \end{bmatrix} \in K_\phi$ if and only if $x_+ \in H^2$, $g \in L^2$ and

$$x_- := \bar\phi x_+ + \bar\psi g \in H^2_-.$$

Alternatively, we can characterize K_ϕ as pairs $\begin{bmatrix} x_+ \\ x_- \end{bmatrix}$ such that

$$x_\pm \in H^2_\pm \quad \text{and} \quad g := \frac{x_- - \bar\phi x_+}{\bar\psi} \in L^2.$$

It looks natural to hope that the pairs

$$\begin{bmatrix} x_+ \\ x_- \end{bmatrix} = \begin{bmatrix} \psi y_+ \\ \bar\psi y_- \end{bmatrix}, \quad y_\pm \in H^2_\pm \tag{1.13}$$

form a dense set in K_ϕ (recall ψ is an outer function). The corresponding g,

$$g = y_- - \frac{\bar\phi\psi}{\bar\psi} y_+ = y_- + f_0 y_+,$$

for sure belongs to L^2 and the element of K_ϕ is of the form

$$\begin{bmatrix} \psi y_+ \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ y_- + P_- f_0 y_+ \end{bmatrix} \oplus \begin{bmatrix} \psi y_+ \\ P_+ f_0 y_+ \end{bmatrix} \in \begin{bmatrix} 0 \\ H^2_- \end{bmatrix} \oplus \hat H. \tag{1.14}$$

However, in fact,

Theorem 1.3. *A function ϕ is regular if and only if the vectors of the form* (1.14) *form a dense set in K_ϕ, or what is the same,*

$$\check H_\phi := \operatorname{clos}_{H^2(\mathbb{C}^2)} \left\{ \begin{bmatrix} \psi y_+ \\ P_+ f_0 y_+ \end{bmatrix} : y_+ \in H^2 \right\} = \hat H_\phi. \tag{1.15}$$

Moreover, (1.15) *holds as soon as*

$$P_+ \bar t \begin{bmatrix} \phi(t) \\ \psi(t) \end{bmatrix} = \begin{bmatrix} \frac{\phi(t)}{t} \\ \frac{\psi(t)-\psi(0)}{t} \end{bmatrix} \in \check H_\phi. \tag{1.16}$$

For a proof see e.g. [13].

A trivial consequence is the following

Proposition 1.4. *Let*

$$\frac{1}{1-|\phi|^2} \in L^1. \tag{1.17}$$

Then ϕ is regular.

Indeed, we put $x(t) = \frac{\phi(t)}{t\psi(t)} \in H^2$ and we get

$$\begin{bmatrix} \psi x \\ P_+ f_0 x \end{bmatrix} = \begin{bmatrix} \frac{\phi(t)}{t} \\ P_+ \frac{1}{t}(\psi - 1/\bar\psi) \end{bmatrix} = \begin{bmatrix} \frac{\phi(t)}{t} \\ \frac{\psi(t)-\psi(0)}{t} \end{bmatrix},$$

since $\frac{1}{t} 1/\bar\psi \in H^2_-$.

One of the main goal of this note is to discuss: is it possible to give a better then (1.17) sufficient condition in terms of the absolute value of ϕ?

We have to point out on a nice result that was obtain in [12], see also [11, 13]. It was shown that there is no necessary and sufficient condition of regularity of ϕ in terms of the absolute value $|\phi|$.

Theorem 1.5. *Let $\phi \in H^\infty$ satisfies* (1.4). *Then there exists an inner function Φ such that $\phi\Phi$ is regular.*

2. Condition on modulus $|\psi|$ which ensures regularity of ϕ but which is weaker than $1/\psi \in H^2$.

We want to see some non-trivial conditions on $|\phi|$ that guarantee that ϕ is regular. By non-trivial we understand any condition different from

$$\frac{1}{|\psi|^2} = \frac{1}{1-|\phi|^2} \in L^1\,. \tag{2.1}$$

We denote by h the outer function with modulus

$$|h|^2 := \frac{1}{1-|\phi|^2}\,.$$

In other words

$$h = \frac{1}{\psi}\,.$$

Outer h always exists by the assumption (1.4) on ϕ. It is *not* in H^2 throughout this section because we are looking for "non-triviality".

Theorem 2.1. *Suppose* $|h|^2 := \frac{1}{1-|\phi|^2} \notin L^1(\mathbb{T})$. *Suppose also that*

$$\liminf_{N\to\infty} (\int_{|h|\le N} |h|^4 dm)(\int_{|h|>N} (\log|h|)^4 dm) = 0\,, \tag{2.2}$$

Then ϕ *is regular.*

Corollary 2.2. *Suppose* $|h|^2 := \frac{1}{1-|\phi|^2} \notin L^1(\mathbb{T})$. *Suppose also that*

$$\liminf_{N\to\infty} N^4 \int_{|h|>N} (\log|h|)^4 dm) = 0\,, \tag{2.3}$$

Then ϕ *is regular.*

Let us explain a bit assumptions (2.2), (2.3). It is easy to to fulfill them if there exists a sequence $N_n \to \infty$ such that

$$|\{\zeta\in\mathbb{T} : |h| > N_n\}|(N_n \log N_{n+1})^4 \le \frac{1}{n^2}\,. \tag{2.4}$$

On the other hand, this is easily reconcilable with the following condition which guarantees non-triviality:

$$|\{\zeta\in\mathbb{T} : |h| = N_{n+1}\}|(N_{n+1})^2 \ge (n+1)^2\,. \tag{2.5}$$

In fact, to have both (2.5) and (2.4) one can define $|h|$ to be step-function having values $N_n, n = 1,2,3,...$ on sets having measures $\frac{n^2}{N_n^2}$ (this gives (2.5)) , and choose N_n going to infinity extremely fast to have firstly $\sum_{k=n+1}^\infty \frac{k^2}{N_k^2} < \frac{2(n+1)^2}{N_{n+1}^2}$ and secondly

$$\frac{(n+1)^2}{N_{n+1}^2}(N_n \log N_{n+1})^4 \le \frac{1}{2n^2}\,.$$

Then (2.4) follows.

Remark. Conditions (2.2), (2.3) are of course the condition just on $|\phi|$, or, which is the same, on $|\psi|$.

Now we will prove Theorem 2.1. We are grateful to A. Aleksandrov whose idea is used in the proof.

Proof. We need to use (2.2) to prove the existence of H^2 functions v_n such that

$$\psi v_n \to \frac{\phi(\zeta)}{\zeta} \text{ in } H^2 \text{ and}$$

$$-P_+\left(\frac{\bar\phi\psi}{\bar\psi} v_n\right) \to \frac{\psi(\zeta)-\psi(0)}{\zeta} \text{ in } H^2\,.$$

Notice that if we would have $\frac{1}{|\psi|^2} = |h|^2 \in L^1$ then we could have taken

$$v_n = \frac{\phi(\zeta)}{\zeta\psi(\zeta)}$$

which would have been functions in H^2 in this case. But we have exactly opposite case: $\frac{1}{|\psi|^2} = |h|^2 \notin L^1$.

Notice that to satisfy the above relationships it is enough to build $g_n \in H^2$ such that

$$g_n h \in H^2 \ \ \forall n\,, \tag{2.6}$$

and such that

$$g_n \to 1 \text{ in } H^2 \text{ and } -P_+\left(\frac{|\phi|^2\bar\zeta}{\bar\psi} g_n\right) \to \frac{\psi(\zeta)-\psi(0)}{\zeta} \text{ in } H^2\,. \tag{2.7}$$

In fact, having g_n like that we put $v_n = \frac{\phi(\zeta)}{\zeta} h g_n$. Then $v_n \in H^2$ by (2.6). And these v_n satisfy two conditions mentioned above because of (2.7).

Now let us write

$$-P_+\left(\frac{|\phi|^2\bar\zeta}{\bar\psi} g_n\right) = -P_+(\bar\zeta\frac{g_n}{\bar\psi}) + P_+(\bar\zeta\psi g_n) =: I_n + II_n\,.$$

Now

$$II_n = \frac{\psi g_n - (\psi g_n)(0)}{\zeta} \to \frac{\psi-\psi(0)}{\zeta}$$

in H^2 because $g_n \to 1$ in H^2, ψ is from H^∞ and backward shift operator is bounded in H^2.

So the only thing we need now is to construct g_n such that (2.6) holds, $g_n \to 1$ in H^2, and

$$-I_n = P_+(\bar\zeta\bar h g_n) \to 0 \tag{2.8}$$

in H^2.

To have all this it is enough to have

$$P_+(\bar\zeta\bar h g_n) \to 0 \text{ in } H^2\,; \ \ g_n h \in H^2\,; \ \ g_n \to 1 \text{ in } H^2\,. \tag{2.9}$$

Let us fix a sequence $N_n \to \infty$, put

$$\phi_n = \begin{cases} \log|h|, & |h| > N_n \\ 0, & |h| \le N_n \end{cases}$$

Here are our

$$g_n := e^{-(\phi_n + i\widetilde{\phi_n})}$$

In fact, obviously,

$$\|P_+(\bar{\zeta}\bar{h}g_n)\|_2^2 \leq \|\bar{h}e^{-(\phi_n+i\widetilde{\phi_n})} - \bar{h}e^{-(\phi_n-i\widetilde{\phi_n})}\|_2^2\,. \tag{2.10}$$

And, of course, $|g_n||h| \leq N_n$.

Now we have $\|\bar{h}e^{-(\phi_n+i\widetilde{\phi_n})} - \bar{h}e^{-(\phi_n-i\widetilde{\phi_n})}\|_2^2 = 4\|he^{-\phi_n}\sin\widetilde{\phi_n}\|_2^2$. In conjunction with (2.10) this gives (we also use that $|\sin x| \leq |x|$ and the definition of ϕ_n)

$$\frac{1}{4}\|P_+(\bar{h}g_n)\|_2^2 \leq \|he^{-\phi_n}\widetilde{\phi_n}\|_2^2 \leq \int_{|h|\leq N_n} |h|^2|\widetilde{\phi_n}|^2 + \int_{\mathbb{T}} |\widetilde{\phi_n}|^2 =: J_1 + J_2. \tag{2.11}$$

To estimate J_1 we write (using the boundedness of the harmonic conjugation operator $\widetilde{\cdot}$ in $L^4(\mathbb{T})$)

$$\begin{aligned} J_1 \leq \int_{|h|\leq N_n} |h|^2|\widetilde{\phi_n}|^2 \leq (\int_{|h|\leq N_n} |h|^4 dm)^{\frac{1}{2}} (\int_{\mathbb{T}} |\widetilde{\phi_n}|^4 dm)^{\frac{1}{2}} \leq \\ C\,(\int_{|h|\leq N_n} |h|^4 dm)^{\frac{1}{2}} (\int_{\mathbb{T}} |\phi_n|^4 dm)^{\frac{1}{2}} = \\ C\,(\int_{|h|\leq N_n} |h|^4 dm)^{\frac{1}{2}} (\int_{|h|>N_n} |\log|h||^4 dm)^{\frac{1}{2}}\,. \end{aligned} \tag{2.12}$$

In particular, if (2.2) holds, there exists a sequence of numbers $N_n \to \infty$ such that the last expression tends to zero. Thus $J_1 \to 0$.

Now let us estimate J_2.

$$\begin{aligned} J_2 = \int_{\mathbb{T}} |\widetilde{\phi_n}|^2 \leq (\int_{\mathbb{T}} |\widetilde{\phi_n}|^4\, dm)^{\frac{1}{2}} \leq C\,(\int_{\mathbb{T}} |\phi_n|^4\, dm)^{\frac{1}{2}} = \\ C\,(\int_{|h|>N_n} |\log|h||^4\, dm)^{\frac{1}{2}} \leq C\,(\int_{|h|\leq N_n} |h|^4 dm)^{\frac{1}{2}} (\int_{|h|>N_n} |\log|h||^4 dm)^{\frac{1}{2}}\,. \end{aligned} \tag{2.13}$$

The last inequality holds for large N_n. In fact, for large N_n integral $\int_{|h|\leq N_n} |h|^4 dm$ is as large as we wish because we assumed that $\int_{\mathbb{T}} |h|^2 dm = \infty$.

Therefore, if (2.2) holds then $J_2 \to 0$. Going back to (2.10) we see that we proved Theorem 2.1.

□

3. Strong Regularity

The strong regularity means that ϕ is regular and in addition $\|\Gamma\| < 1$, where $\Gamma = \Gamma_{f_0}$. In other words, ϕ is strongly regular if and only if ϕ is regular and the operator $(I - \Gamma^*\Gamma)$ is invertible.

The following theorem is a combination the Helson-Szegö, Hunt-Muckenhoupt-Wheeden, and Adamyan-Arov-Krein Theorems (AAK), for a matrix generalization see e.g. [5, 6].

Theorem 3.1. *Function ϕ is strongly regular if and only if*

$$\frac{1-|\phi|^2}{|1-\phi|^2} = w \in A_2. \tag{3.1}$$

Note that the left hand side in (3.1) being a positive harmonic function in the unit disc ($=\mathrm{Re}\frac{1+\phi}{1-\phi}$) is equal to the harmonic extension of an A_2 weight on the circle.

Proof. Let ϕ is regular and $\|\Gamma_{f_0}\| < 1$. Consider the symbol f_1, which corresponds to the choice $\mathcal{E} = 1$ in (1.7) (recall f_0 corresponds to $\mathcal{E} = 0$). It is of the form

$$f_1 = \frac{\bar{g}}{g}, \quad g := \frac{1-\phi}{\psi}$$

and we have $\Gamma = \Gamma_{f_1}$. By the Adamyan-Arov-Krein Theorem (AAK), see Remark 3.2,

$$\frac{1}{\psi\psi(0)} = (I - \Gamma^*\Gamma)^{-1}1 \in H^2. \tag{3.2}$$

Thus $g \in H^2$ and $\|\Gamma_{\frac{\bar{g}}{g}}\| < 1$. By the Helson-Szegö and Hunt-Muckenhoupt-Wheeden Theorems, see e.g. [16],

$$|g|^2 = \frac{|1-\phi|^2}{1-|\phi|^2} \in A_2.$$

Conversely, from (3.1) we conclude

$$\frac{1+\phi}{1-\phi} = w + i\widetilde{w}\,, \tag{3.3}$$

as before $\widetilde{w}$ stands for the harmonic conjugate of w (the Hilbert transform on the circle).

Then of course

$$\frac{1-\phi}{1+\phi} = \frac{1}{w+i\widetilde{w}}\,,$$

and so

$$\frac{1-|\phi|^2}{|1+\phi|^2} = \mathrm{Re}\frac{1}{w+i\widetilde{w}} = \frac{w}{w^2+\widetilde{w}^2}\,. \tag{3.4}$$

Let us derive that

$$\frac{|1+\phi|^2}{1-|\phi|^2} \in L^1 \tag{3.5}$$

on the unit circle. Indeed,

$$\int_{\mathbb{T}} \frac{|1+\phi|^2}{1-|\phi|^2} = \int_{\mathbb{T}} w + \int_{\mathbb{T}} \frac{\widetilde{w}^2}{w} \le \|w\|_1 + Q_{1/w}\|w\|_1 < \infty\,,$$

where $Q_{1/w}$ stands for the norm of the Hilbert transform from $L^2_{1/w}$ to itself, which is finite as $w \in A_2$.

Combine (3.5) with a simple remark that (3.1) implies $\frac{|1-\phi|^2}{1-|\phi|^2} \in L^1(\mathbb{T})$. Add these two relations and obtain

$$\frac{1}{1-|\phi|^2} \in L^1(\mathbb{T})\,. \tag{3.6}$$

We know that this is sufficient for being regular.

Finally, by the converse statement in the Helson-Szegö and Hunt-Muckenhoupt-Wheeden Theorems we have $\|\Gamma\| = \|\Gamma_{f_1}\| = \|\Gamma_{\frac{\bar{g}}{g}}\| < 1$, if $|g|^2 \in A_2$. The latter is exactly (3.1). $\square$

Remark 3.2. Let us comment (3.2) from the point of view of regularity. We still assume that $\|\Gamma\| < 1$, that is $(I - \Gamma^*\Gamma)^{-1}1$ has the direct meaning. Then it is easy to check that the vector

$$\check{k} := \begin{bmatrix} \psi(I-\Gamma^*\Gamma)^{-1}1 \\ P_+ f_0 (I-\Gamma^*\Gamma)^{-1}1 \end{bmatrix} \psi(0) \tag{3.7}$$

is the reproducing kernel in $\check{H}^2_\phi$, see (1.15). Indeed,

$$\langle \begin{bmatrix} \psi y_+ \\ P_+ f_0 y_+ \end{bmatrix}, \check{k}\rangle = \langle (I-\Gamma^*\Gamma)y_+, (I-\Gamma^*\Gamma)^{-1}1\rangle \psi(0) = y_+(0)\psi(0).$$

Also, it is evident that the reproducing kernel of $\hat{H}^2_\phi$ is $\hat{k} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, i.e.:

$$\langle \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \hat{k}\rangle = x_1(0), \quad \forall \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \hat{H}^2_\phi.$$

Thus, $\hat{H}_\phi = \check{H}_\phi$ implies $\hat{k} = \check{k}$, and the equality of the first components is precisely (3.2).

Generally, in AAK theory, for a regular ϕ the following formula holds

$$\frac{1}{\psi(\zeta)\psi(0)} = \lim_{r\downarrow 1}((rI - \Gamma^*\Gamma)^{-1}1)(\zeta),$$

so the last function is not necessary in H^2.

3.1. **Less than one.** The previous proof exploited a lot of AAK theory in its part that proves 3.1 from strong regularity, and we wish to give a more direct proof for the reader who is not so familiar with this subtle material.

The second proof.

First we need an AAK lemma, which can be found by the reader in [16] or extracted from AAK papers from our references list (however we provide the proof for the sake of completeness).

Lemma 3.3. *Let $F \in L^\infty$ and $d(F, H^\infty) < 1$. Then the coset $F + H^\infty$ contains a function $\frac{\bar{h}}{h}$, where $|h|^2 \in A_2$. In particular, this coset contains the unimodular function v such that the Toeplitz oerator T_v is invertible.*

Proof. Let $d(f, H^\infty)$ denote the distance between a function $f \in L^\infty$ and the subspace H^∞ of bounded holomorphic functions in the unit disc. Let H^∞_0 denote bounded holomorphic functions in the unit disc that vanish at zero. Given such an F consider $d(\bar{z}F, H^\infty) = d(F, H^\infty_0)$. Two cases may happen. Suppose first that $d(\bar{z}F, H^\infty) = 1$. Then operator Hankel operator $H_{\bar{z}F} := P_-(\bar{z}F\cdot) : H^2 \to H^2_-$ attains its norm. In fact, $\|H_{\bar{z}F}\|_{ess} \le d(\bar{z}F, \bar{z}H^\infty) = d(F, H^\infty) < 1 = d(\bar{z}F, \bar{z}H^\infty) = \|H_{\bar{z}F}\|$. This is just classical Nehari's theorem (see [16]), and $\|\cdot\|_{ess}$ means the norm modulo compact operators (essential norm). If the essential norm of the operator A in the Hilbert space is strictly less than its norm, then A attains its norm. See [16] Ch VII again, or just notice that we can reduce our statement to self-adjoint operators by considering A^*A (and polar decomposition $A = U(A^*A)^{1/2}$). But if the essential norm of the self adjoint operator is strictly smaler than its norm, it means that its norm is a maximal eigenvalue od finite multiplicity (spectral theorem), and, thus, the operator attains its norm. Now let our Hankel operator attain its norm 1 at vector $H \in H^2, \|H\|_2 = 1$. Denote $\bar{G}_0 = H_{\bar{z}F}H$. Denote by u a function in the coset $\bar{z}F, H^\infty$ of $\|u\|_\infty = 1$. It always exists by obvious compactness argument.

Then

$$1 = \|\bar{G}_0\|_2 = \|H_{\bar{z}F}H\| = \|H_u H\|_2 = \|P_(uH)\| \leq \|uH\|_2 \leq \|H\|_2 = 1\,.$$

This means of course that $|u| = 1$ almost everywhere on the circle and that uH is antianalytic (that is $P_+(uH) = 0$).

Therefore,

$$uH = P_-(uH) = \bar{G}_0\,.$$

We conclude that two H^2-functions H and G_0 have the same modulus a. e. on the unit circle. Write $H = S_1 h$, $G_0 = zS_2 h$ their inner-outer factorizations (h is an outer function here).

Then we obtain

$$\frac{\bar{h}}{S_0 h} = u \in \bar{z}F + H^\infty\,,$$

where $S_0 = zS_1S_2$, $h \in H^2$. Consider $S = zS_1S_2$ and

$$v := zu = \frac{\bar{h}}{Sh} \in F + H^\infty\,.$$

We want to prove now that Toeplitz operator T_v is invertible. From the fact that $d(v, H^\infty) = d(F, H^\infty < 1$ and from Nehari's theorem we know that $\|H_v\| < 1$. But $|v| = 1$ a. e. on the unit circle and then $T_v^* T_v = I - H_v^* H_v$. Therefore $\|H_v\| < 1$ means that T_v is bounded from below (that is it is left-invertible). To prove that it is invertible it is sufficient to prove that its adjoint has only trivial kernel. Let $R \in Ker T_v^* = T_{\bar{v}}$. Then

$$T_{\bar{v}}R = 0 \Rightarrow P_+(\frac{h}{\bar{S}\bar{h}}R) = 0 \Rightarrow \frac{h}{\bar{S}\bar{h}}R = \bar{z}\bar{r}\,,$$

where $r \in H^2$. Then $hR = \bar{z}\bar{S}\bar{h}\bar{r}$, the left hand side being from H^1, and the right hand side being from H^1_-. The intersection being zero we conclude that $R = 0$.

So we get v, $v := zu = \frac{\bar{h}}{Sh} \in F + H^\infty$ such that T_v is invertible. By Helson-Szegö theorem (see [16], Ch. VII) we conclude that $v = \frac{\bar{h}}{h}$, where $|h|^2 \in A_2$.

Now we need to consider the second case: $d(\bar{z}F, H^\infty) < 1$. We denote $f := \bar{z}F$ and consider the function $\tau(c) := d(f + c\bar{z}, H^\infty)$. We know that $\tau(0) < 1, \tau(\infty) = \infty$ and τ is obviously continuous. So we can find c_0 such that for $f + c_0\bar{z} =: \bar{z}\Phi$

$$d(\bar{z}\Phi, \bar{z}H^\infty) = d(f + c_0\bar{z}, \bar{z}H^\infty) = d(f, H^\infty) < 1\,,$$

$$d(\bar{z}\Phi, H^\infty) = d(f + c_0\bar{z}, H^\infty) = 1\,.$$

The we proceed exactly as in the first case by using the fact that the last two relationships imply that operator $H_{\bar{z}\Phi}$ attains its norm. We will find unimodular $u \in f + c_0\bar{z} + H^\infty$ such that

$$u = \frac{\bar{g}}{z\Theta g}\,,$$

where Θ is inner and g is outer from H^2. Therefore, $v := zu$ will be in coset $zf + H^\infty = F + H^\infty$ and will have the form $u == \frac{\bar{g}}{\Theta g}$ wit the same Θ and g.

Again as in the first case $\|H_v\| = d(F, H^\infty) < 1$ (Nehari's teorem) ensures that T_v is left-invertible. And exactly as before we prove that $T_v^* = T_{\bar{v}}$ has a trivial kernel. Hence T_v is invertible and we conclude once again by Helson-Szegö theorem (see [16], Ch. VII) that $v = \frac{\bar{h}}{h}$, where $|h|^2 \in A_2$. AAK lemma is proved. $\square$

It is easy to finish the second proof of our theorem. Let ϕ be strongly regular. This means that it is reguar and so $f_0 := -\frac{\bar\phi\psi}{\bar\psi}$ is such that

$$f_0 + H^\infty = f_0 + \frac{\psi^2 e}{1-\phi e}\,,$$

where e runs over the unit ball of H^∞. But strong regularity means also that $\|H_{f_0}\| < 1$. Lemma 3.3 means that there exists an outer h such that $\frac{\bar h}{h} \in f_0 + H^\infty$ and $|h|^2 \in A_2$. We gather:

$$\frac{\bar h}{h} = -\frac{\bar\phi\psi}{\bar\psi} + \frac{\psi^2 e}{1-\phi e}$$

for some e from the unit ball of H^∞. Then a. e. on the circle

$$\left| -\frac{\bar\phi\psi}{\bar\psi} + \frac{\psi^2 e}{1-\phi e}\right| = 1\,.$$

But the left hand side is

$$\left|\frac{e-\phi}{1-\phi e}\right|,$$

and we conclude that$e$ is an inner function. Then

$$\frac{\bar h}{h} = -\frac{\bar\phi\psi}{\bar\psi} + \frac{\psi^2 e}{1-\phi e} = \frac{\psi\overline{1-\phi e}}{\bar\psi(1-\phi e)}\,.$$

Denote $g_e := \frac{1-\phi e}{\psi}$. Then we just got

$$\frac{\bar h}{h} = e\frac{\bar g_e}{g_e}\,.$$

But $|1/g_e|^2 = \frac{1-|\phi e|^2}{|1-\phi e|^2} \in L^1$, see (1.11), and so $1/g_e \in H^2$ (g_e is obviously an outer function). We then immediately conclude from the last display equality that $g = c\cdot h$ with some constant c. In fact, we have

$$e\frac{h}{g_e} = \frac{\bar h}{\bar g_e}\,,$$

where the left hand side belongs to H^1 and the right hand side belongs to H^1_-. So both expressions are just constants. And $e = e^{ir}$, $g_e = c\cdot h$, as it has been promised. Therefore, $\frac{|1-e^{ir}\phi|^2}{1-|\phi|^2} \in A_2$.

Now we use Theorem 3.5 proved in the Appendix to conclude that $\frac{|1-\phi|^2}{1-|\phi|^2} \in A_2$. We finished the proof that regularity implies 3.1.

The following criteria was proposed in [6].

Theorem 3.4. *ϕ is strongly regular if and only if the matrix weight*

$$W := \begin{bmatrix} 1 & \phi \\ \bar\phi & 1\end{bmatrix} \tag{3.8}$$

satisfies matrix A_2 condition.

Matrix A_2 condition was found in [21]. Let us notice that Theorem 3.1 has the following curious corollary.

Theorem 3.5. *Let w be such that (3.1) holds. Consider a new positive harmonic function given by*

$$w_{e^{ic}} := \frac{1-|\phi|^2}{|1-e^{ic}\phi|^2}. \tag{3.9}$$

The positive harmonic function $w_{e^{ic}}$ does not have singular part in its Herglotz representation. It absolutely continuous part on the circle (also called $w_{e^{ic}}$) is uniformly in A_2.

It is absolutely trivial from the point of view of Theorem 3.4: If matrix function W is in matrix A_2 then the following matrix is obviously also in matrix A_2:

$$W_c := \begin{bmatrix} e^{ic/2} & 0 \\ 0 & e^{-ic/2} \end{bmatrix} \begin{bmatrix} 1 & \phi \\ \bar\phi & 1 \end{bmatrix} \begin{bmatrix} e^{-ic/2} & 0 \\ 0 & e^{ic/2} \end{bmatrix}$$

is just $W_c = \begin{bmatrix} 1 & \phi_c \\ \bar\phi_c & 1 \end{bmatrix}$, where $\phi_c = e^{ic}\phi$.

The matrix A_2 condition for the specific weight (3.8) can be given as the following scalar condition (this is a slight modification of the condition given in [6]).

Lemma 3.6. *A weight of the form (3.8) satisfies A_2 if and only if*

$$\sup_I \frac{1}{|I|}\int_I \frac{|\phi-\langle\phi\rangle_I|^2 + (1-|\langle\phi\rangle_I|^2)}{1-|\phi|^2}\, dm < \infty, \tag{3.10}$$

where for an arc $I \subset \mathbb{T}$ we put

$$\langle\phi\rangle_I := \frac{1}{|I|}\int_I \phi\, dm. \tag{3.11}$$

Proof. The matrix weight $\begin{bmatrix} 1 & \phi \\ \bar\phi & 1 \end{bmatrix}$ is in A_2 implies

$$\begin{aligned} \frac{1}{|I|}\int_I \frac{\begin{bmatrix} 1 & -\phi \\ -\bar\phi & 1 \end{bmatrix}}{1-|\phi|^2} dm \leq & C \begin{bmatrix} 1 & \langle\phi\rangle_I \\ \langle\bar\phi\rangle_I & 1 \end{bmatrix}^{-1} \\ = & C\left\{ \begin{bmatrix} 1 & 0 \\ \langle\bar\phi\rangle_I & \sqrt{1-|\langle\phi\rangle_I|^2} \end{bmatrix} \begin{bmatrix} 1 & \langle\phi\rangle_I \\ 0 & \sqrt{1-|\langle\phi\rangle_I|^2} \end{bmatrix} \right\}^{-1}, \end{aligned} \tag{3.12}$$

or

$$\frac{1}{|I|}\int_I \frac{\begin{bmatrix} \frac{1-|\phi|^2+|\phi-\langle\phi\rangle_I|^2}{(\overline{\langle\phi\rangle_I-\phi})\sqrt{1-|\langle\phi\rangle_I|^2}} & (\langle\phi\rangle_I-\phi)\sqrt{1-|\langle\phi\rangle_I|^2} \\ & 1-|\langle\phi\rangle_I|^2 \end{bmatrix}}{1-|\phi|^2} dm \leq C, \tag{3.13}$$

which is equivalent to (3.10). $\square$

Remark 3.7. The latter form (3.10) of condition (3.1) makes indeed evident that this condition is invariant with respect to the rotation $\phi \mapsto \phi e^{ic}$. In fact, the condition is stable with respect to an arbitrary fraction–linear transform

$$\phi(\zeta) \mapsto e^{ic}\frac{\phi(\zeta)-a\zeta}{1-\bar a\phi(\zeta)/\zeta}, \quad |a|<1,\ c\in\mathbb{R}.$$

Since a proof of Theorem 3.4 is fairly simple and short we give it here.

Proof. First we note that (3.10) in case $I = \mathbb{T}$ has the form

$$\int_{\mathbb{T}} \frac{|\phi|^2 + 1}{1 - |\phi|^2}\, dm < \infty, \tag{3.14}$$

therefore by Proposition 1.4 such ϕ is regular.

Now, recall that a weight is in A_2 if and only if (there exists $Q > 0$)

$$\langle W^{-1} P_+ X, P_+ X\rangle \le Q \langle W^{-1} X, X\rangle,\ \forall X \in L^2_{W^{-1}}. \tag{3.15}$$

Note that

$$H^2 \to W \begin{bmatrix} 0 \\ H^2 \end{bmatrix} \quad H^2_- \to W \begin{bmatrix} H^2_- \\ 0 \end{bmatrix}$$

are unitary embedding in $L^2_{W^{-1}}$.

The orthogonal complement to their sum (an alternative definition of K_ϕ) consists of the vectors of the form

$$K_\phi = \{X = \begin{bmatrix} x_+ \\ x_- \end{bmatrix},\ x_\pm \in H^2_\pm\}.$$

The vectors

$$X = \begin{bmatrix} \psi x_+ \\ \bar\psi x_- \end{bmatrix}$$

evidently belongs to K_ϕ. They are *dense* in K_ϕ if and only if $\hat H_\phi = \check H_\phi$, i.e. ϕ is regular, Theorem 1.3.

Now we calculate the quadratic forms (3.15) for the test–vectors

$$X = \begin{bmatrix} \psi x_+ \\ \bar\psi x_- \end{bmatrix} + W \begin{bmatrix} y_- \\ y_+ \end{bmatrix}, \quad x_\pm \in H^2_\pm,\ y_\pm \in H^2_\pm,$$

We get

$$\begin{aligned}
\langle W^{-1} X, X\rangle =& \langle \frac{\begin{bmatrix} 1 & -\phi \\ -\bar\phi & 1 \end{bmatrix}}{1 - |\phi|^2} \begin{bmatrix} \psi x_+ \\ \bar\psi x_- \end{bmatrix}, \begin{bmatrix} \psi x_+ \\ \bar\psi x_- \end{bmatrix}\rangle + \langle \begin{bmatrix} 1 & \phi \\ \bar\phi & 1 \end{bmatrix} \begin{bmatrix} y_- \\ y_+ \end{bmatrix}, \begin{bmatrix} y_- \\ y_+ \end{bmatrix}\rangle \\
=& \langle \begin{bmatrix} 1 & \bar f_0 \\ f_0 & 1 \end{bmatrix} \begin{bmatrix} x_+ \\ x_- \end{bmatrix}, \begin{bmatrix} x_+ \\ x_- \end{bmatrix}\rangle + \langle y_-, y_-\rangle + \langle y_+, y_+\rangle \\
=& \langle \begin{bmatrix} 1 & \Gamma^* \\ \Gamma & 1 \end{bmatrix} \begin{bmatrix} x_+ \\ x_- \end{bmatrix}, \begin{bmatrix} x_+ \\ x_- \end{bmatrix}\rangle + \langle y_-, y_-\rangle + \langle y_+, y_+\rangle.
\end{aligned}$$

Since

$$P_+ X = \begin{bmatrix} \psi x_+ \\ 0 \end{bmatrix} + \begin{bmatrix} \phi \\ 1 \end{bmatrix} y_+,$$

we get

$$\begin{aligned}
\langle W^{-1} P_+ X, P_+ X\rangle &= \langle \begin{bmatrix} 1 \\ -\bar\phi \end{bmatrix} \frac{x_+}{\bar\psi} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y_+, \begin{bmatrix} \psi x_+ \\ 0 \end{bmatrix} + \begin{bmatrix} \phi \\ 1 \end{bmatrix} y_+\rangle \\
&= \langle x_+, x_+\rangle + \langle y_+, y_+\rangle.
\end{aligned}$$

Therefore for such vectors (3.15) is equivalent to

$$\|x_+\|^2 + \|y_+\|^2 \le Q\{\langle (I - \Gamma^*\Gamma) x_+, x_+\rangle + \|\Gamma x_+ + x_-\|^2 + \|y_-\|^2 + \|y_+\|^2\},$$

which is evidently equivalent to $(I - \Gamma^*\Gamma) > \epsilon I$ with $\epsilon > 0$.

Thus, A_2 implies the regularity and the bound $\|\Gamma\| < 1$. Conversely, if ϕ is regular and $\|\Gamma\| < 1$ then (3.15) holds on a dense set, and hence $W \in A_2$. $\square$

It is interesting that a similar condition on an arbitrary symbol of the Hankel operator guaranties invertibility of $(I - \Gamma^*\Gamma)$.

Proposition 3.8. *Let f satisfies the A_2-kind condition*

$$\sup_I \frac{1}{|I|}\int_I \frac{|f - \langle f\rangle_I|^2 + (1 - |\langle f\rangle_I|^2)}{1 - |f|^2}\, dm < \infty, \tag{3.16}$$

where

$$\langle f\rangle_I := \frac{1}{|I|}\int_I f\, dm. \tag{3.17}$$

Then $\|\Gamma\| < 1$ for $\Gamma x := P_- f x$.

Proof. We have

$$\langle \begin{bmatrix} 1 & \bar f \\ f & 1 \end{bmatrix} P_+X, P_+X\rangle \le Q\langle \begin{bmatrix} 1 & \bar f \\ f & 1 \end{bmatrix} X, X\rangle.$$

Put here

$$X = \begin{bmatrix} x \\ -\Gamma x \end{bmatrix},\ x \in H^2.$$

We get

$$\langle x, x\rangle \le Q\langle (I - \Gamma^*\Gamma)x, x\rangle.$$

□

Appendix.

We wish to prove Theorem 3.5 without relating to strongly regular functions in the sense of Arov-Dym, namely, to give a direct proof of this curious result.

Let us recall that we start with $w \in A_2$. We extend it into the disc by harmonicity, get a positive harmonic function and represent it–as usual–in the form

$$\frac{1 - |\phi|^2}{|1 - \phi|^2} = w\,, \tag{3.18}$$

where ϕ is holomorphic in the disc and of H^∞-norm at most 1. Such functions ϕ and positive harmonic functions are in one to one correspondance by (3.18).

Let $\phi_c := e^{ic}\phi\,, \psi_c := e^{ic}\psi\,, c \in \mathbb{R}$. Consider a new positive harmonic function given by

$$w_c := \frac{1 - |\phi_c|^2}{|1 - \phi_c|^2}\,, \tag{3.19}$$

Theorem 3.9. *If $w \in A_2$, and $\phi \in H^\infty$, $\|\phi\|_\infty \le 1$ is given by* (3.18)*, then a positive harmonic function w_c from* (3.19) *does not have singular part in its Herglotz representation. Its absolutely continuous part on the circle (also called w_c) is uniformly in A_2.*

Proof. We already saw that (3.18) implies

$$\frac{1}{1 - |\phi|^2} \in L^1(\mathbb{T})\,.$$

Then if we put $g_c := \frac{1-\phi_c}{\psi}$ we get that outer function g_c always is such that $g_c \in H^2$.

Assume for a moment that we can prove the folowing:

$$\Gamma_{\frac{\bar g_c}{g_c}} = e^{-ic}\Gamma_{\frac{\bar g_0}{g_0}}\,. \tag{3.20}$$

The last operator has norm less than 1. In fact, $|g_0|^2$ is given to be in A_2 by (3.18). And Then by Helson-Szegö's theorem we have this estimate of the norm of Hankel operator, again see[16]. If the Hankel operator with unimodular symbol has norm strictly lees than 1, then the Toeplitz operator with the same symbol is obviously bounded from below.

We conclude that for each real c Toeplitz operator $T_{\frac{\bar{g_c}}{g_c}}$ is left invertible, and, combining this with the fact that $g_c \in H^2$, we conclude that its adjoint has a trivial kernel. Then $T_{\frac{\bar{g_c}}{g_c}}$ is invertible. But unimodular symbols of invertible Toeplitz operators are $\frac{\bar{h}}{h}$ with $|h|^2 \in A_2$, see [16]. Now $1/g_c = \frac{\psi}{1-\phi_c}$ and $1/|g_c|^2 = \frac{1-|\phi_c|^2}{|1-\phi_c|^2} = \mathrm{Re}\frac{1+\phi_c}{1-\phi_c}$ in the unit disc. The latter function has positive real part, and so $1/|g_c|^2 \in L^1(\mathbb{T})$. As $1/g_c$ is outer, we conclude that $1/g_c \in H^2$.

Using that $1/g_c \in H^2$ and writing $\frac{\bar{g_c}}{g_c} = \frac{\bar{h}}{h}$ we conclude that $g_c = const \cdot h$. Therefore, $\frac{|1-\phi_c|^2}{1-|\phi|^2} = |g_c|^2 \in A_2$.

To finish the proof of our Theorem 3.9 we are left to prove (3.20). For that we want to prove

$$\frac{\bar{g_c}}{g_c} \in e^{-ic}\frac{\bar{g_0}}{g_0} + H^\infty \,. \tag{3.21}$$

Let us write a chain of equalities:

$$\frac{\bar{g_0}}{g_0} = \frac{\overline{1-\phi}}{1-\phi} \cdot \frac{\psi}{\bar{\psi}} = \frac{(|\psi|^2 + |\phi|^2)\psi - \bar{\phi}\psi}{(1-\phi)\bar{\psi}} = -\frac{\bar{\phi}\psi}{\bar{\psi}} + \frac{\psi^2}{1-\phi}\,.$$

Similarly,

$$\frac{\bar{g_c}}{g_c} = e^{-ic}\frac{\overline{1-\phi_c}}{1-\phi_c} \cdot \frac{\psi_{\frac{c}{2}}}{\bar{\psi}_{\frac{c}{2}}} = -e^{-ic}\left[\frac{\bar{\phi}_c\psi_{\frac{c}{2}}}{\bar{\psi}_{\frac{c}{2}}} + \frac{\psi_{\frac{c}{2}}^2}{1-\phi_c}\right] =$$

$$-e^{-ic}\frac{\bar{\phi}\psi}{\bar{\psi}} + \frac{\psi^2}{1-\phi_c}\,.$$

Comparing these two equalities we obtain:

$$\frac{\bar{g_c}}{g_c} = e^{-ic}\frac{\bar{g_0}}{g_0} + \left(\frac{\psi^2}{1-\phi_c} - e^{-ic}\frac{\psi^2}{1-\phi}\right).$$

Both functions in the brackets are from H^∞. In fact, they are obviously from Smirnov class. Also their boundary values are at most 2 by absolute value: from the above chains of equalities one can see that they are simply sums of two unimodular functions each.

$\square$

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